11/4-18 Föreläsning 6

Recap to wrap-up the modeling part

Input-output models (ODE ↑ degree)

State-space models (matrix diff. eq., 1st order)

$\tilde{x}(t) = Tx(t)$

Solved the state eq.

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau$$

for $\dot{x}(t) = Ax(t) + Bu(t)$, x_0

$$y(t) = Cx(t) + Du(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)} Bu(\tau)d\tau$$

Stability

eigenvalues for A

$\forall\ i = 1 \dots n$, $Re\{\lambda_i\} \leq 0$

Lyaponov / bounded
stable

$Re\{\lambda_i\} < 0 \Rightarrow \lim_{t \to \infty} x(t) = 0$ asymptotic stable

Controllable

Can we reach/control each and every state with the input?

$R(A,B) = [B \mid AB \; \dots \; A^{n-1}B]$

$\dim(x) = n$

reachability matrix

rank $R = n \Rightarrow$ u can reach every x

Today :

1) Observability

—— " —— modeling done

2) What does it stand for PID?

3) How to tune a PID compensator?

Time
Laplace] domain specs.

4) Nyquist stability I

1) Observability :

With $\{u, y\}$ can we observe the x?

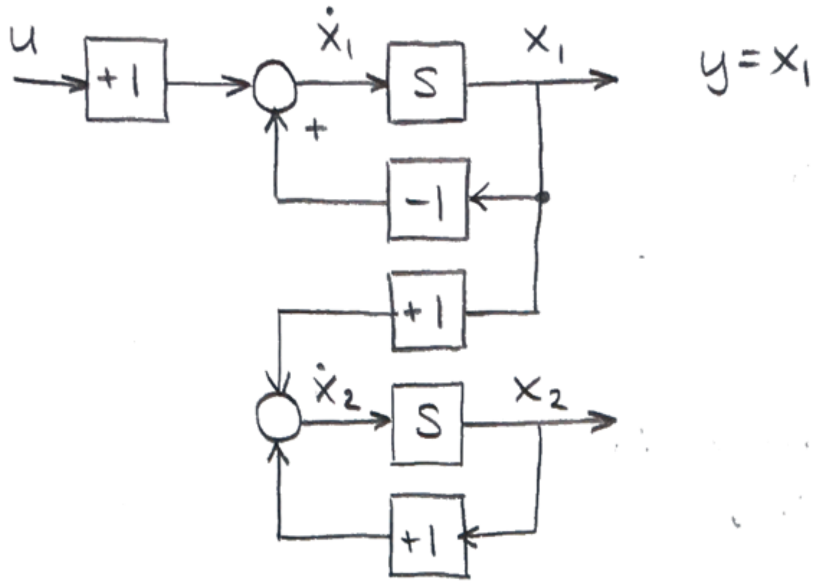
e.g. $$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [1 \;\; 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$\dim(x) = 2$

Draw it's block diagram!

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$



$x_1 \longrightarrow y$ $\qquad$ $x_2(t)$ will never be in $y(t)$

$x_2 \not\longrightarrow y$

$x_2 \not\longrightarrow x_1 \longrightarrow y$

Def. The LTI state-space model is observable if for any T $x(T)$ can be observed by the $\{u, y\}$ sequence over $[0, T]$.

How to check?

Theorem (Kalman observability condition)

A continuous time LTI model is observable iff rank $\sigma = n$, where

$$\sigma(C, A) = \sigma = \begin{bmatrix} C \\ \hline CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \qquad \sigma \in \mathbb{R}^{n \times n} \quad \text{(for SISO)}$$

e.g. $\quad A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = [1 \ \ 0]$

Observable?

$n = 2 \Rightarrow CA^{n-1} = CA$

$$\sigma = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \end{bmatrix}$$

To check full rank property:

$\det \sigma = 1 \cdot 0 - (-1) \cdot 0 = 0$

$\Downarrow$

rank $\sigma < 2 \Rightarrow$ not observable

Note if $\det \sigma \neq 0 \Rightarrow \sigma$ has full rank.

Theorem

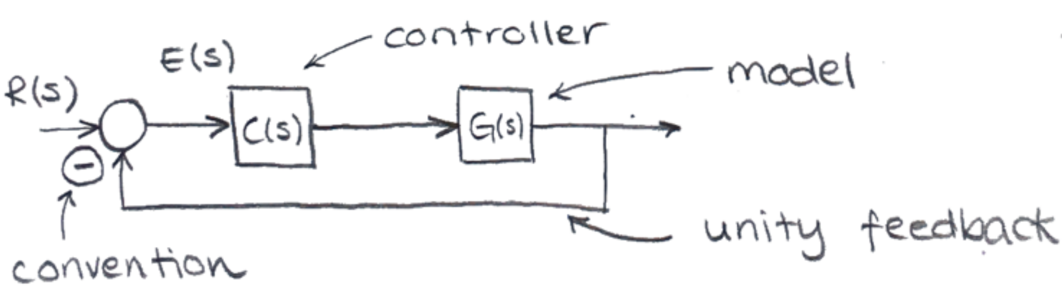
An LTI model is called minimal order both controllable and observable at the same time.

Minimality: $u \longrightarrow y$ relation is given with the least element of x.

Input / output stability implies internal stability.

2) What does it stand for PID?

Def. PID is a structured, dynamic and output feedback controller



$E(s) = R(s) - Y(s)$

Basic form for PID | controller
regulator
compensator

$$C(s) = K_p + \frac{K_i}{s} + K_d s$$

proportional (P) constant — integrator (I) dynamic — differentiator (D) dynamic

$$u(t) = \mathcal{L}^{-1}\{u(s)\} = K_p e(t) + K_i \int_0^t e(\tau)\,d\tau + K_d \frac{de(t)}{dt}$$

output feedback because only R(s) and Y(s) are required for it!

Roles of:

P - constant / fast speed up closed loop

I - cummulate tracking error / slow / get rid of error $\lim_{t\to\infty} e = 0$

D - "look ahed" feat future.

e.g. virus epidemy model

$$\ddot{I}(t) + a_1 \dot{I}(t) + a_2 I(t) = b u(t)$$

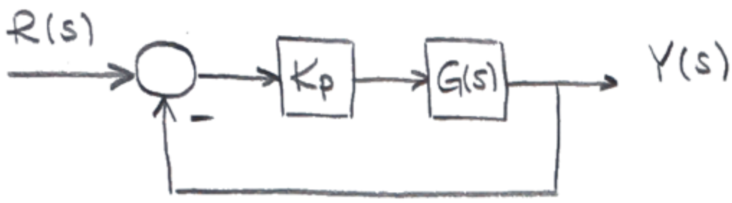
$I(t)$ - # of infected species

$u(t)$ - immunitation

Find a stable controlled epidemic behaviour!

$y(t) = I(t)$; $C(s) = K_p$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b}{s^2 + a_1 s + a_2}$$



$$G_c(s) = \frac{Y(s)}{R(s)} = \frac{G(s) \cdot C(s)}{1 + G(s)C(s)}$$ (see L4, L3)

after creating the closed-loop.

$$L(s) = C(s)G(s) = \frac{K_p \cdot b}{s^2 + a_1 s + a_2}$$

loop transfer

$f(s) = 1 + L(s)$ ← loop ratio return

$$G_c(s) = \frac{L(s)}{L(s)+1} = \frac{K_p b}{s^2 + a_1 s + \underbrace{a_2 + K_p b}}$$

K_p only influences the a_2 ⇒ p_1, p_2 can <u>not</u> be arbitrary.

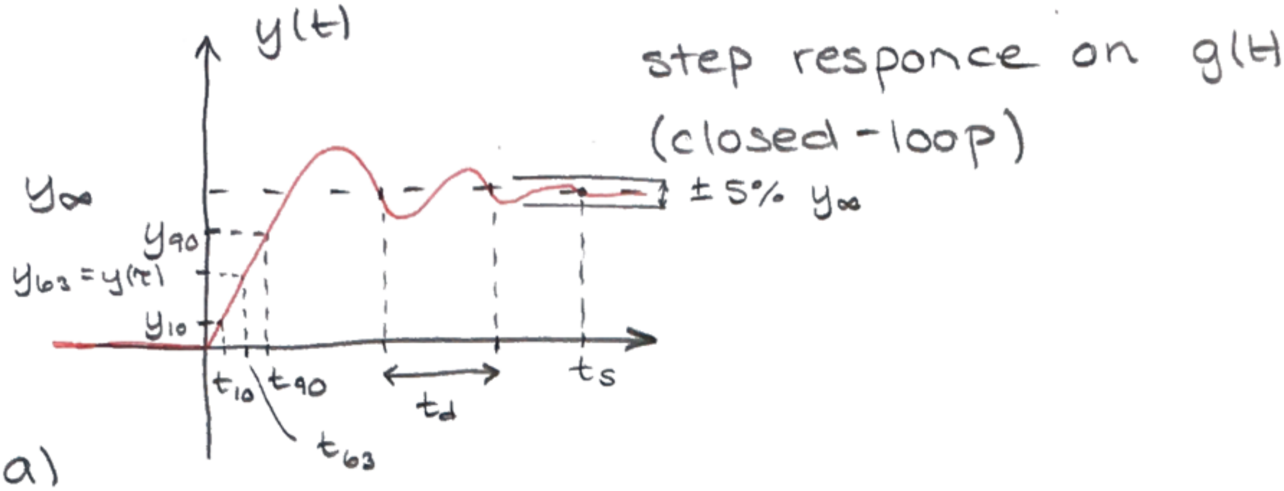
changed ⇒ It may remain unstable under

$u(t) = K_p (r(t) - y(t))$

$e(t)$ can be unbounded depending on (a_1, a_2)

3) How to tune a PID controller?

Time domain specifications



a) y_∞ steady-state output

$\Rightarrow$ $e_\infty = r_\infty - y_\infty$ stationary tracking error.

b) $y_{10} = y(t_{10})$ – t_{10} is when you reach 10% of y_∞

$t_r = t_{90} - t_{10}$ rise time

c) Equivalent time constant τ

$y_{63} = y(\tau)$ 63% of y_∞ reached.

d) t_d = damping time

e) t_s = settling time

$0{,}95 \leq y(t) \leq y_\infty \; 1{,}05$

enters into and stay there $\Rightarrow$ the $y(t) \approx y_\infty$

f) t_p = peak value for $y(t)$

$\Rightarrow$ largest $y(t_p)$

g) overshoot $\sigma = \dfrac{y(t_p) - y_\infty}{y_\infty}$ [%]

	t_r	σ	t_s	e_∞
K_p		↑		↓
K_i		↑		0
K_d		↓		–

homework: fill in table.

↑ increase, ↓ decrease, 0 removes, - no effort

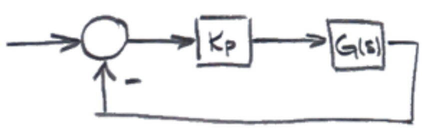
Method for tuning PID

1) Henristics < Graph / Transfer function

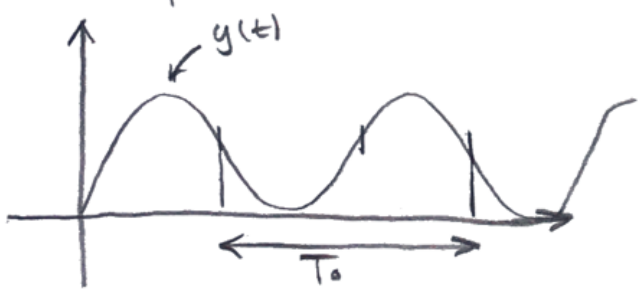
2) Guaranteed stability

1.1 Graphical: Zegler-Nichols
You aim finding K_p, K_i, K_d?

Step 1)
closed-loop. Assumption, $G(s)$ can be stabilized by $C(s) = K_p$.
$K_i = 0$; $K_d = 0$ and use K_p



from $k_p \ll 1$ start increasing $K_p = K_0$



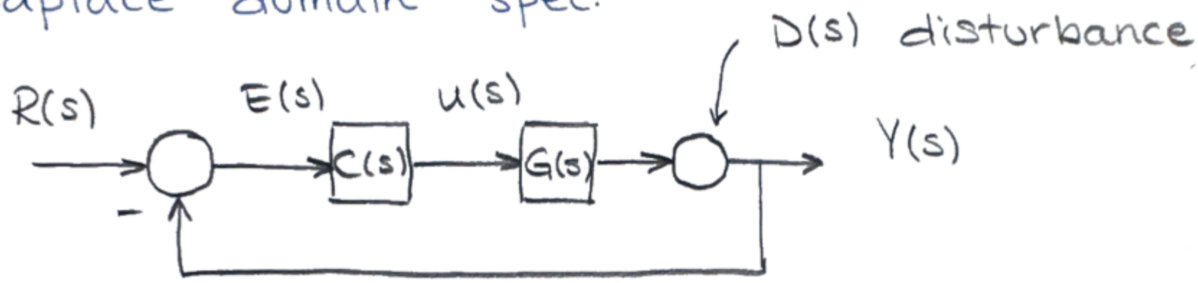
T_0 - period

with K_0 we are at the stability limit.

step 2)

ZN	K_p	$1/K_i$	$1/K_d$
P	$0{,}5 K_0$	–	–
PI	$0{,}45 K_0$	$T_0/1{,}2$	–
PID	$0{,}6 K_0$	$T_0/2$	$T_0/8$

Laplace domain spec.



$$D(s) \equiv 0$$

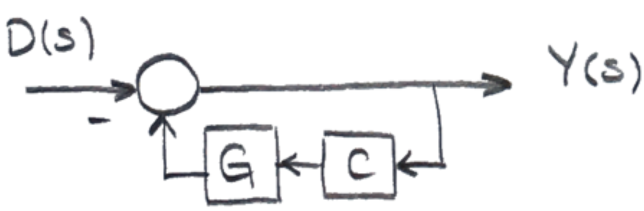
$$G_c(s) = \frac{Y(s)}{R(s)}$$

$$Y(s) = \frac{L(s)}{1+L(s)} \cdot R(s) = T(s) \cdot R(s)$$

complamentary sensitivity

how good is the tracking?

$R(s) \equiv 0 \qquad Y(s)/D(s)$



$$Y(s) = \frac{L(s)}{1+L(s)} D(s) \left[\approx \frac{\text{forward}}{\text{loop ratio}} \right]$$

$Y(s) = S(s) \cdot D(s)$

$S(s)$ - sensitivity

No matter what you select for $C(s)$

Theorem

$T(s) + S(s) = 1$

Guaranteed tracking

$e_\infty = 0$

$$E(s) = 1 \cdot R(s) - Y(s) \underset{\text{theorem}}{=} (\cancel{T(s)} + S(s))\, R(s) - \cancel{T(s)} \cdot R(s)$$

$Y(s) = T(s) \cdot R(s)$

$$E(s) = S(s) \cdot R(s) = \frac{1}{L(s)+1} \cdot R(s)$$

$$\lim_{t \to \infty} e(t) \Big|_{R(s) = \frac{1}{s}} = \lim_{t \to 0} s \cdot \frac{1}{L(s)+1} \cdot \frac{1}{s}$$

$$e_\infty = \lim_{s \to 0} \frac{1}{L(s)+1}$$

Suppose $L(s) = \frac{1}{s} \cdot L_0(s)$ — remaining rational transfer fcn.

$$e_\infty = \lim_{s \to 0} \frac{s}{s + L_0(s)} = 0$$

If the loop has integrator it eliminates $e(t)$.

4) Nyquist theory I

$$L(s) = \frac{b(s)}{a(s)}$$ – loop behaviour

$$T(s) = \frac{L(s)}{L(s)+1}$$ – closed loop

1) understand what is the diff. between $L(s) \sim T(s)$

2) Use conditions on $L(s)$ to get stability for closed-loop.