9/4-18 Föreläsning 5

1) From state-space to transfer function
2) Solution to the LTI state-space
3) Stability
4) Controllability
5) Observability

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1) From state-space to transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b(s)}{a(s)}$$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \qquad x_0 = 0$$

$$\mathcal{L}(\dot{x}(t) = Ax(t) + Bu(t))$$

$$(sI - A)X(s) = BU(s)$$

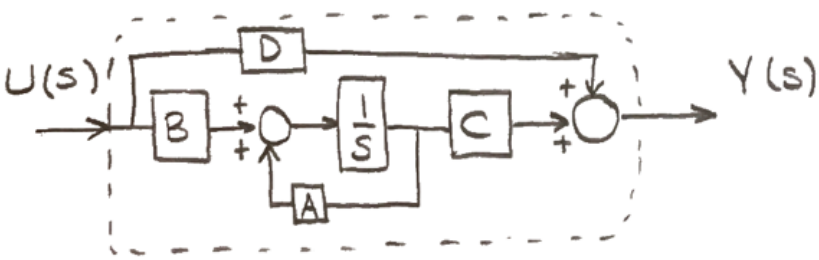
$$X(s) = BU(s)(sI-A)^{-1}$$

$$\mathcal{L}(y(t) = Cx(t) + Du(t))$$

$$Y(s) = CX(s) + DU(s)$$

$$Y(s) = (C(sI-A)^{-1}B + D)U(s)$$

$$G(s) = C(sI-A)^{-1}B + D$$



TF above is parametrized by (A, B, C, D).

$$\left.\begin{array}{l} x(t)(A, B, C, D) \\ \tilde{x}(t)(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \end{array}\right\} \text{non-unique } \boxed{\text{state-space models}}$$

Theorem 1:

given two state-space forms (A, B, C, D) and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with $\tilde{x}(t) = Tx(t)$ $(\exists\, T^{-1})$ then $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$, $\tilde{D} = D$.
$(T \in \mathbb{R}^{n \times n}$, non-singular$)$

T - similarity state transformation matrix

Proof

$$\dot{x}(t) = \frac{d}{dt}(T^{-1}\tilde{x}(t)) = T^{-1}\dot{\tilde{x}}(t) = A\underbrace{x(t)}_{T^{-1}\tilde{x}(t)} + Bu(t) =$$

$$\Rightarrow I\dot{\tilde{x}}(t) = TAT^{-1}\tilde{x}(t) + TBu(t) \qquad /\text{premultiply}$$

$$\dot{\tilde{x}}(t) = \underbrace{\tilde{A}}_{TAT^{-1}}\tilde{x}(t) + \underbrace{\tilde{B}}_{TB}u(t)$$

$$y(t) = Cx(t) + Du(t)$$
$$y(t) = CT^{-1}\tilde{x} + Du(t)$$
$$y(t) = \tilde{C}\tilde{x}(t) + Du(t)$$
$$\tilde{C} = CT^{-1}$$

Note, T does not influence the direct feedthrough term i.e. $\tilde{D} = D$.

2) Solve the state-equation

$x_0 = x(t=0)$ solve the problem

$\dot{x}(t) = Ax(t) + Bu(t)$

$y(t) = Cx(t) + Du(t)$

Theorem 2:

For an autonomous LTI state equation

$\dot{x}(t) = Ax(t)$, $x_0 \neq 0$

$x(t) = e^{At} \cdot x_0$

where $e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots$

matrix exponential.

autonomous / no input - closed-loop system model

Theorem 3:

Given x_0

$\dot{x}(t) = Ax(t) + Bu(t)$

$y(t) = Cx(t) + Du(t)$

then

$$x(t) = \underbrace{e^{At} x_0}_{\text{homogeneous solution}} + \underbrace{\int_0^t e^{A(t-\tau)} Bu(\tau) d\tau}_{\text{particular inhomogeneous solution part } (u(t))}$$

Proof

$\dot{x}(t) = Ax(t) + Bu(t)$, premultiply e^{-At}

$$\underbrace{e^{-At} \cdot \dot{x}(t) - e^{-At} \cdot A \cdot x(t)}_{f \cdot \dot{g} + \dot{f} \cdot g} = e^{-At} Bu(t)$$

$$e^{At} A = (I + At + A^2 \frac{t^2}{2!} + \dots) A = e^{At}$$

$$= (A + A^2 t + A^3 \frac{t^2}{2!} + \dots) = A(I + At + A^2 \frac{t^2}{2!} + \dots)$$

$$e^{-At} \cdot A = Ae^{-At}$$

$$\frac{d(e^{-At} x(t))}{dt} = e^{-At} \cdot Bu(t)$$

$$e^{-At} x(t) - I \cdot x_0 = \int_0^t e^{-A\tau} B\, u(\tau) d\tau$$

$$x(t) = e^{At} x_0 + \underbrace{\int_0^t e^{A(t-\tau)} B\, u(\tau) d\tau}_{\int_0^t g(t-\tau) u(\tau) d\tau}$$

How to get e^{At}?

e^{At} - state transition matrix

In the time domain: develop Taylor based approx $e^{At} \cong I + At + A^2 \frac{t^2}{2!}$

$$\exp\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} t\right) \neq \begin{bmatrix} e^{a_{11}t} & e^{a_{12}t} \\ e^{a_{21}t} & e^{a_{22}t} \end{bmatrix}$$

generally speaking

If A is in a diagonal form

$$A_d = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \quad \forall \ \lambda_n \neq \lambda_j \in \mathbb{C}$$

e.g. $$x(t) = e^{A_d \cdot t} x_0 = e^{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix} t} x_0 = \sum_{j=1}^{n} \underbrace{e^{\lambda_j \cdot t}}_{\text{basis function}} x(j)$$

$x(j) = e^{\lambda_j t} x_0(j)$

$y(t) = Cx(t) = [1\ 1\ 1 \dots]\, x(t)$

$y(t) = \sum_{j=1}^{n} e^{\lambda_j t} x_0(j)$

We solved the state equation analytically:

x_0, e^{At}, $\int e^{A(t-\tau)} Bu(\tau) d\tau$

$\Rightarrow x(t)$

What kind of other analytical properties does it have (stability, controllable, etc.)?

3) Stability

Internal mode stability.

Theorem 4:

Given a continous LTI state-space (autonomous) system model, $\dot{x} = Ax(t)$, x_0.
The model is Lyapunov / asymptot stable if eigenvalues of $A (\det(\lambda I - A) = 0)$ are having

non-positive real parts $Re(\lambda_n) \leq 0$

or

strictly negative real parts $Re(\lambda_i) < 0$

$\forall\ i = 1 \dots n$

Lyapunov - bounded output

Asymptotic says $\lim_{t \to \infty} x(t) = 0$

This is an internal stability concept.

Eg $G(s) = \frac{\cancel{(s-1)}}{(s+1)\cancel{(s-1)}}$ stable ?

a) $G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+1} \Rightarrow p = -1$

Input - outputwise it is STABLE!

b) $G(s) = \frac{s-1}{s^2-1} \longrightarrow$ create controller conon. f.

$$\begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow eig(A)$$

$\dim(x) = 2 \longleftarrow s^2 \qquad \lambda_{1,2} = \pm 1$

$a(s) = s^2 - 1 = s^2 + a_1 s + a_2$

Internally the system model is UNSTABLE

zero - pole cancellation hides internal unstable mode.

Internal $\longrightarrow$ input output

Stability $\not\longleftarrow$ stability

generally speaking not true.

eig(A) $\not\rightleftarrows$ poles of $a(s)=0$ are not always the same.

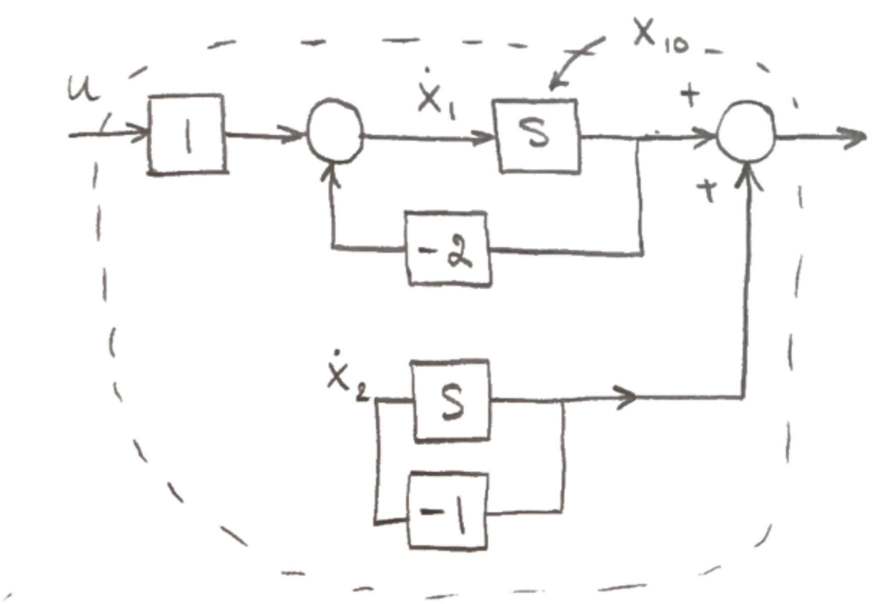
Note, if the state-space is controllable and observable (obs + cont. = minimal state space) then internal stability coincides with input-output

4) Controllability

e.g. Block diagram of $\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$

$y(t) = [1 \;\; 1]\, x(t)$

$\dim(x) = 2$, SISO



$y(x)$

neihter directly nor ($u \not\to x_2$)

indirectly $x_2(t)$ can be influenced by $u(t)$.

$u \to x_1 \not\to x_2$

x_2 can not be controlled/reached out.

Def. (Reachability $x_0 = 0 \longrightarrow x_f \neq 0$ arbitrary).

Given (A, B, C, D), the LTI system model is said reachable if $\forall$ $x_f = x(t_f)$ terminal state there exists a bounded input $u(t)$, t_f (finite) to travel from $x_0 \longrightarrow x_f$.

5) Controllability

Def. (Controllability $x_0 \neq 0 \longrightarrow x_f = 0$)

The system model (A, B, C, D) is controllable if for any initial condition $x_0 \neq 0$ there $\exists$ a bounded input sequence and finite time that steers the trajectory to $x_f = 0$.

Controllability and reachability only depends on (A, B)

Theorem 5: A continous LTI state-space model is controllable / reachable if

$\text{rank } R = \dim(x)$

$R = [B \,\vdots\, A \cdot B \ldots A^{n-1} B]$

Note, this is the same Kalman rank condition.

e.g.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\dim(x) = 2 = n$

$$R = [B \,\vdots\, AB \ldots \underbrace{A^{2-1} B}_{AB}] = [B \,\vdots\, AB] = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \vdots \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] =$$

$$= \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right] \qquad (\det(R) = 0)$$

$2 = \dim(x) \neq \text{rank } R = 1$

$\Rightarrow$ is not reach. / cont.