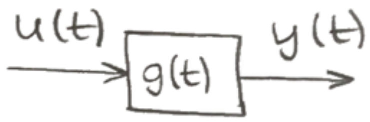
28/3-18

Föreläsning 4



g(t) - system models

y(t)?

interconnection of models

stable?

So far: we had Input/Output models

1) Closed-loop stability
2) From Input/Output models to state-space (initial) model.
3) Variants of statespace models (controllable, observeable, digonal)
4) State-space models to transfer fcns.

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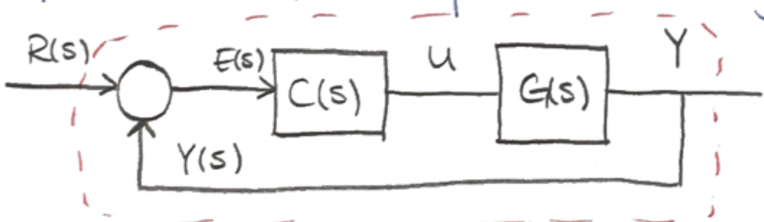
1) Closed-loop stability



$$G(s) = \frac{Y(s)}{R(s)}$$

closed-loop

$$G_c(s) = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

$$L(s) = C(s)G(s) = G(s)C(s)$$

loop transfer function

$$G_c(s) = \frac{L(s)}{1+L(s)} = \frac{b_c(s)}{a_c(s)}$$

$$\left(G(s) = \frac{b(s)}{a(s)} \right)$$

$$a_c(s)\Big|_{p_i} = 0 \qquad \forall \quad Re(p_i) \leq 0$$

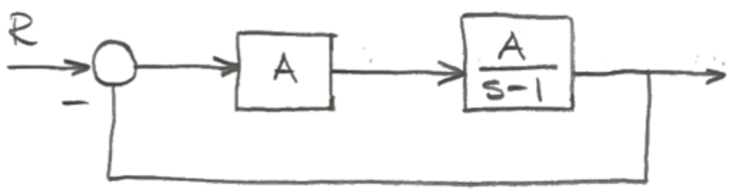
E. g. $\quad G(s) = \dfrac{A}{s-1}$

$\infty > A > 0$, proportional controller.

Find A such that the closed-loop is stable!

- open is unstable

  $p = +1$



Note, negative and unity output feedback applies.

$$G_c = \frac{L(s)}{1+L(s)} = \frac{A\left(\frac{1}{s-1}\right)}{1+\frac{A}{s-1}} = \frac{A}{s-1+A}$$

$\infty > A \geqslant 1$

$\longrightarrow \quad P_c \leqslant 0 \quad$ (stable) $\Big]$ a) BIBO $A \geq 1$

asymptotic

b) stat $A > 1$

2) From Input/Output models to State-space (initial) models.

We had models that used the I/O relationship to create a system model.

I/O models

Given $\{y(0), \dot{y}(0), \quad y^{(n)}(0)\}$

$(u(0), \dot{u}(0), \quad u^{(m)}(0)\}$, an ode

u(+ derivatives) $\longrightarrow$ y(+ derivatives)

internal variables of a system model (state)

Two stage modelling concept.

E.g.

damping — spring koeff

$$m\ddot{y}(t) + k_D\dot{y}(t) + k_S y(t) = u(t)$$

$y(t)$ is position, $u(t)$ is force state-space model?

select

$x_1(t) = \dot{y}(t)$ ← speed

$x_2(t) = y(t)$ ← position

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \text{vector valued } x \in \mathbb{R}^2$$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \ddot{y}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -\frac{k_D}{m} x_1(t) - \frac{k_S}{m} x_2(t) + \frac{1}{m} u(t) \\ x_1(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -k_D/m & -k_s/m \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\text{state vector}} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Generally:

$\dot{x}(t) = Ax(t) + Bu(t)$ – (i) dynamic eqn
$y(t) = Cx(t) + Du(t)$ – (ii) matrix diff. eqn
output eqn. (static)

State-space models

$x \in \mathbb{R}^n \quad u, y \in \mathbb{R}^1$, $\dim(A) = [n \times n]$
$\dim(B) = [n \times 1]$
$\dim(C) = [1 \times n]$
$\dim(D) = [1 \times 1]$ (scalar)

(i) stage one
(ii) stage two
} two stage modelling concept

A, B, C, D are constant coeff. matrixes. Instead of an ODE for I/O we have a matrix diff. eqn. (first order)!

Def. (state)

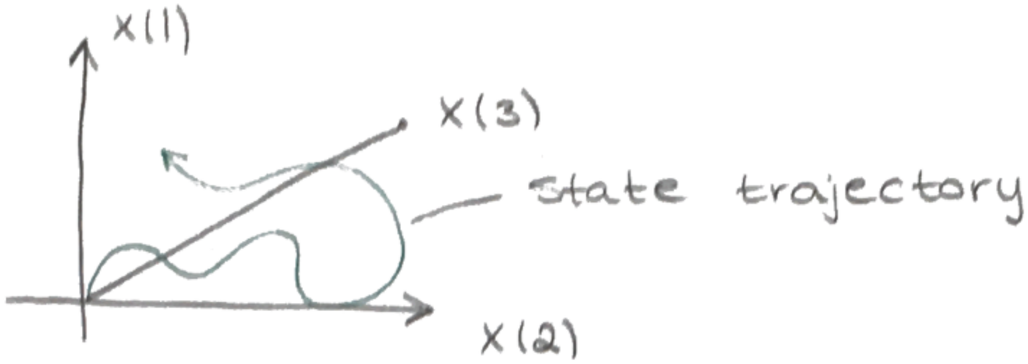
The vector valued quantity $x(t) \in \mathbb{R}^n$ is said a state, internal memory. It describes the variation (inner) of energi / speed /

/ position / entropy / temp etc.

Def (state-space)

The state vector $x(t)$ belongs to a state-space $X \subset \mathbb{R}^n$

Usually X is bounded



Can we transform transfer fcns to state-space?

E.g. $$G(s) = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2} = \frac{b(s)}{a(s)}$$

Find a state space model!

$$\frac{Y(s)}{U(s)} = G(s) = \frac{b(s)}{a(s)}$$

$$Y(s) = b(s) \underbrace{a^{-1}(s)\, U(s)}_{Z(s)} = b(s) Z(s)$$

dummy
intermediate
variable/signal

$$Y(s) = b_0 s \cdot Z(s) + b_1 Z(s)$$

$$U(s) = a(s) Z(s) \Rightarrow s^2 Z(s) + a_1 s Z(s) + a_2 Z(s)$$

$$\mathcal{L}^{-1}(1/s) = y(t) = b_0 \dot{z} + b_1 z(t)$$

$$\mathcal{L}^{-1}(u(s)) = u(t) = \ddot{z}(t) + a_1 \dot{z}(t) + a_2 z(t)$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dot{z}(t) \\ z(t) \end{bmatrix}$$

suggestion!

Order selection for state-space look at the peak diff $\ddot{z}(t)$

$\Rightarrow 2 \qquad \dot{x}_2(t) = x_1(t)$ (by random pick).

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{z}_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ \end{bmatrix} u(t)$$

$$y(t) = [b_0 \ b_1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Remark. $x = \begin{bmatrix} \dot{z} \\ z \end{bmatrix}$; what if $\tilde{x} = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}$ is this good?

Yes!

From transfer fcn to state-space the transform is not unique.

$\Big[$ A state-space is never unique to describe I/O relation! $\Big]$

Note, there exist some dedicated / canonical / standard forms for state-space models.

3) Variants of state-space models

Controllable form

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n}$$

$$\dot{x} = \underbrace{\begin{bmatrix} -a_1 & -a_2 & \dots & a_n \\ 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u(t)$$

$n = \dim(x)$

$y(t) = \underbrace{[b_0 \; b_1 \; \dots \; b_m]}_{C} x(t)$

$D = 0$

Note, $x(t)$ does not have physical meaning (composite state variable)

- This form exists iff the state-space model is controllable (see next lecture)
- Use this form for controller design in state space.

Observer form

Note,

- No physical state-meaning
- Exists iff the state-space is observable (see next lecture)
- Useful if you design observer (Kalman filter)

$A_o = A_c^T$ ← controller canonical

observer canonical form

$B_o = C_c^T$

$C_o = B_c^T$

$\dot{x} = A_o x + B_o u$

$y = C_o x$

HW. find it from (A_c, B_c, C_c)

Note, controller and observer forms and "dual" state space forms. Dualism means transpose model properties.

Diagonal form

Note,

- Not meaningful state values
- decoupling to independent dynamics.
- modal transformation

E.g. $$G(s) = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2}$$

Find $G_1(s) + G_2(s) = G(s)$

Assume $p_1 \neq p_2$ (poles)

$$G(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2}$$

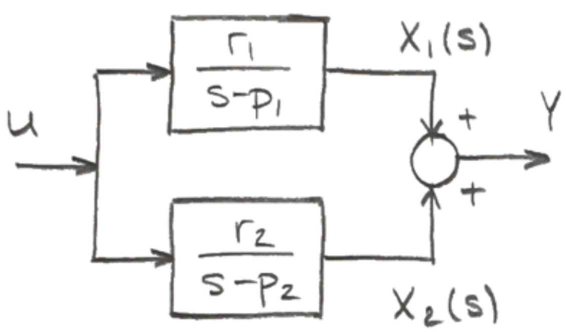
$\{r_1, r_2\}$ - residues

$$r_1 = \lim_{s \to p_1} \cancel{(s-p_1)} \frac{b_0 s + b_1}{\cancel{(s-p_1)}(s-p_2)}$$

$$r_2 = \lim_{s \to p_2} \cancel{(s-p_2)} \frac{b_0 s + b_1}{\cancel{(s-p_2)}(s-p_2)}$$

$$r_1 = \frac{b_0 p_1 + b_1}{p_1 - p_2} \quad ; \quad r_2 = \frac{b_0 p_2 + b_1}{p_2 - p_1}$$

$$Y(s) = \left(\frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} \right) u(s)$$



$$\mathcal{L}^{-1} \begin{cases} (s-p_1) X_1(s) = r_1 \, u(s) \\ (s-p_2) X_2(s) = r_2 \, u(s) \\ Y(s) = X_1(s) + X_2(s) \end{cases}$$

Diagonal state space

$$\dot{x}_1(t) = p_1 x_1(t) + r_1 u(t)$$

$$\dot{x}_2(t) = p_2 x_2(t) + r_2 u(t)$$

$$\dot{x} = \begin{bmatrix} p_1 & & & 0 \\ & p_2 & & \\ & & \ddots & \\ 0 & & & p_n \end{bmatrix} x(t) + \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} x(t)$$

Similarity transformation

$\tilde{x} = Tx$

$\exists\ T^{-1}, \quad T \in \mathbb{R}^{n \times n}$

$$(A, B, C, D) \underset{T}{\xrightarrow{T^{-1}}} (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$$