21/3 - 18 · Föreläsning

- Solution to models convolution int.
- Typical inputs/outputs
  - Impulse
  - Step
- Stability theory

Solution to models: Convolution integral

System models (LTI)

and started solving the model (ODE)

e.g. Given the mechanical system model

$m\ddot{y}(t) = u(t) - k\dot{y}(t) - cy(t)$

$y(t)$ - displacement

$u(t)$ - force

m, k, c are constants mass, damping and stiffness coeffs.

$G(s) = ?$

$\mathcal{L}\{u(t)\} = U(s)$; $\mathcal{L}\{y(t)\} = Y(s)$

$$G(s) = \frac{Y(s)}{U(s)}$$

$\mathcal{L}\{m\ddot{y} = u - k\dot{y} - cy\}_{\text{zero IC}}$

$ms^2 Y(s) + ksY(s) + cY(s) = U(s)$

$$G(s) = \frac{1}{ms^2 + ks + c}$$

Generally speaking (rational)

$$G(s) = \frac{b(s)}{a(s)} = \frac{(b_0 s^m + \ldots + b_m)}{(s^n + a_1 s^{n-1} + \ldots + a_n)}$$

$0 \leq m, n < \infty$

$b(s)\Big|_{s=z_j} = 0$ zero polynomial

$\{z_1 \ldots z_m\}$ <u>are zeros</u>

$a(s)\Big|_{s=p_i} = 0$

pole polynomial (or caracteristics)

$\{p_1 \ldots p_n\}$ <u>are poles</u>

gain

$$G(s) = \frac{b_0 \prod_{j=1}^{m} (s - z_j)}{\prod_{i=1}^{n} (s - p_i)} , \quad \{b_0, \{z_1 \ldots z_m\}, \{p_1 \ldots p_n\}\}$$

$n > m$ - strickly <u>proper</u> transfer func. (TF)
(strickty causal)

$n \geq m$ - proper (causal)

$n < m$ - improper (non-causal)
(anticipating future inputs/values are required)

proper and strickly proper G(s) physically realiseable.

$$\underbrace{\frac{Y(s)}{u(s)} = G(s)}_{} = \frac{b(s)}{a(s)}$$

$$\mathcal{L}^{-1}\{Y(s) = G(s)\cdot u(s)\} = y(t)$$

$$y(t) = \int_{-\infty}^{+\infty} g(\tau)\, u(t-\tau)\, d\tau \quad , \quad g(t) = \mathcal{L}^{-1}\{G(s)\}$$

convolution integral.

$\tau < 0$ we violate causality

$$\boxed{y(t) = \int_{0}^{\infty} g(\tau)\, u(t-\tau)\, d\tau}$$

generic time domain solution to our modeling question.

We assume the system model is in rest

$\forall\, t < 0 \qquad g(t) = 0$

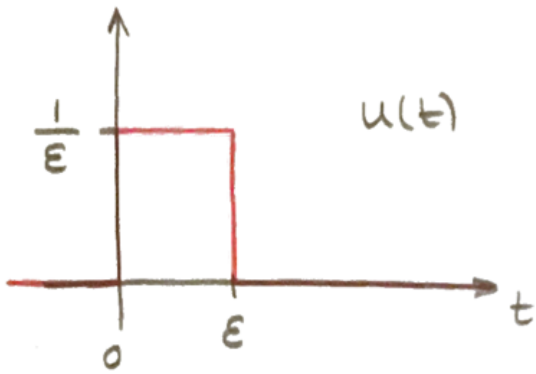
$\qquad\qquad\quad u(t) = 0$

system is timeinvariant we can shift and let it start from $t=0$.

Typical inputs/outputs

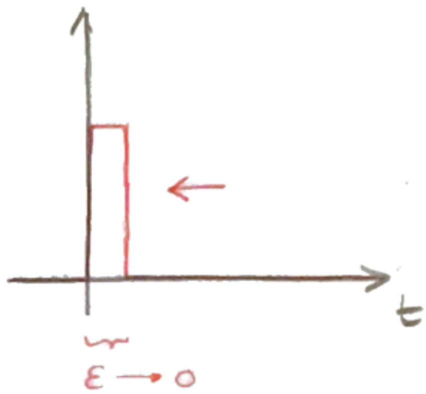
<u>Impulse</u>: fast switch on and off input signal

$$u(t) = \begin{cases} 0 & t < 0 \\ 1/\varepsilon & 0 \le t < \varepsilon \\ 0 & t \ge \varepsilon \end{cases}$$

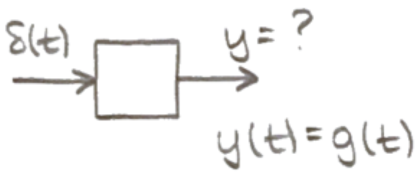


$$\delta(t) = \lim_{\varepsilon \to 0} u(t) \quad \Rightarrow$$

impulse
(Dirac delta)



Big boost to the system at $t=0$



Dirac
delta

$$u(s) = \mathcal{L}\{\delta(t)\} = 1(s)$$

$$Y(s) = G(s) \cdot W(s) = G(s) \cdot 1$$
$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = \mathcal{L}^{-1}\{G(s)\} = g(t)$$

Inputs to LTI models have to be an important property
Input signals have to be bounded

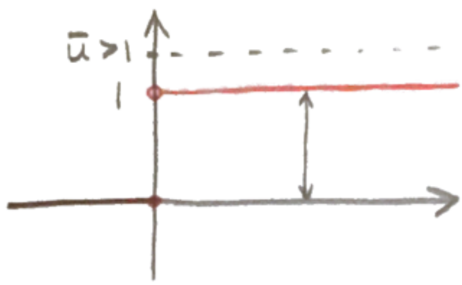
Impulse input is bounded

$$\int_0^\infty \delta(\tau)\,d\tau = 1 = \int_0^t \lim_{\varepsilon\to 0} u(\tau)\,d\tau = \lim_{\varepsilon\to 0}\int_0^{t=\varepsilon} \frac{1}{\varepsilon}\,d\tau =$$

$$= \lim_{\varepsilon\to 0}\left[\tau\cdot\frac{1}{\varepsilon}\right]_0^\varepsilon = 1 - 0 = 1$$

Step input: switch on and stay signal

$$u(t) = \begin{cases} 0 & t<0 \\ 0 & t=0 \\ 1 & t>0 \end{cases}$$



step response

$$Y(s)\Big|_{\mathcal{L}\{u(t)=1(t)\}} = G(s)\cdot U(s) = \frac{G(s)}{s}$$

Is the step input bounded?
Yes, magnitude is bounded!
Relationship between $Y(s)$ and $y(t)$
Laplacetransf. gives answer

In stationary regions (t) you do not need the Laplacetransf.

Final value theorem
(only works for stable models)

$$y_\infty = \lim_{t\to\infty} y(t) = \lim_{s\to 0} s\cdot Y(s)$$

Steady-state $(t \longrightarrow \infty)$ value with $(s \longrightarrow 0)$ zero (frequency) s value.

Initial value theorem

$$y(0) = \lim_{t \to 0} y(t) = \lim_{s \to \infty} s \cdot Y(s)$$

e.g. $G(s) = \frac{6}{s+2}$; what is y_∞ if $u(t) = 1(t)$?

$G(s)$ is stable ;

$$y_\infty = \lim_{s \longrightarrow 0} \cancel{s} \cdot \left(\underbrace{\frac{6}{s+2}}_{G(s)} \cdot \underbrace{\frac{1}{\cancel{s}}}_{u(s)} \right)$$

$$y_\infty = \frac{6}{2} = 3$$

Stability theory

Input signals being bounded will give bounded outputs.

BIBO = bounded inputs bounded outputs

Input bounds can be: energy magnitude ($1, 2, \infty$ - norm signals)

We have 4 equivalent theorem to LTI stability:

Theorem 1:

An LTI model is BIBO stable iff it's impulse response is absolutely integrable, i.e.

$$\int_0^\infty |g(\tau)| \, d\tau < \infty.$$

PROOF

IF (sufficient condition)

if BIBO $|u(t)| < \bar{u}$ then the output will be bounded to.

$$|y(t)| = \left| \int_0^\infty g(\tau) u(t-\tau) d\tau \right| \leq \int_0^\infty |g(\tau)| \cdot |u(t-\tau)| d\tau =$$

convolution integral Δ- inequality

$$= \bar{u} \int_0^\infty |g(\tau)| d\tau$$

BIBO is the model $\bar{u}$ $|y(t)| < \infty \Rightarrow \int_0^\infty |g(\tau)| d\tau < \infty$

Theorem 2:

An LTI system model is BIBO stable iff

$G(s) = b(s)/a(s)$

$a(s)\Big|_{s=p_i} = 0 \quad i = 1 \ldots n \quad \forall i \quad Re(p_i) < 0$

(real values are negative)

G(s) does not have poles at RHP.

Theorem 3:

An LTI model is BIBO stable iff $\lim_{t \to \infty} g(t) = 0$

e.g. $G(s) = \frac{1}{a(s)} = \frac{1}{s^2 + 2s + 4}$

Is this model stable?

(theorem 2)

Find poles

$a(s) = 0 = s^2 + 2s + 4$

(genom pq-formeln)

$s = -1 \pm \sqrt{1^2 - 4} \Rightarrow p_{1,2} = -1 \pm \sqrt{-3}$

$Re(p_{1,2}) < 0 \Rightarrow$

For all bounded input the output will be bounded

e.g $g(t) = \frac{1}{t}$? stable ? (theorem 3)

$$\xrightarrow{u(t)} \boxed{g(t)} \xrightarrow{y(t)}$$

BI BO

stability conditions on $g(t) / G(s)$

Theorem 2; $G(s)$ all poles have to be at LHP

$s \in \mathbb{C}$

Calculate the poles

$a(s)\Big|_{s=p_i} = 0$

Can we find answer to stability w/o computing the poles? Yes.

Routh matrix

e.g. $a(s) = s^2 + 2s + 4 = 0$ $(a_0 s^2 + a_1 s + a_2 = 0)$

Viete's formula

sum and the product of poles/roots will be found by the coeffs. of $a(s)$

$$a_0 = 1 \;;\; a_1 = 2 \;;\; a_2 = 4$$

$$p_1 \cdot p_2 = a_2 / a_0 = 4/1 = 4$$

Both poles are the same signs.

$$p_1 + p_2 = -\frac{a_1}{a_0} = -\frac{2}{1} = -2$$

$\Rightarrow$ Both poles are having negative real values.

Theorem 4:

Suppose $a_i > 0 \quad \forall\, i = 1\, n$

$a(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_n$ then if Routh matrix has only positive first column then the model is stable.

$$a(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + \dots$$

split a's as odd or even

odd - $(a_1, a_3, \dots)$

even - $(a_0, a_2, \dots)$

s^n	a_0	a_2	a_4
s^{n-1}	a_1	a_3	a_5
	b_1	b_2	b_3
	c_1	c_2	c_3

$$b_1 = \frac{a_1 \cdot a_2 - a_0 \cdot a_3}{a_1}$$

$$b_2 = \frac{a_1 \cdot a_4 - a_0 \cdot a_5}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$