23/5-18 Sista föreläsningen

1) Observer design

2) State observer + state feedback

3) Course overview (exam, course PM)

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1) Observer design

Design a dynamic system to reconstruct $x(t)$ based on $u(t)$, $y(t)$, $\{A, B, C, D\}$.

Solution (Luenberger observer)

$\left.\begin{array}{l}\hat{x}(t) - \text{estimated} \\ \hat{y}(t) - \text{—〃—}\end{array}\right\}\begin{array}{l}\text{state}\\ \text{output}\end{array}$

$$\dot{\hat{x}}(t) = \underbrace{A\hat{x}(t) + Bu(t)}_{\text{copy plant}} + \underbrace{K(y(t) - \hat{y}(t))}_{\text{correction (innovation)}}$$

$$\dot{\hat{x}}(t) = (A - KC)\hat{x}(t) + Bu(t) + Ky(t)$$

A-KC is stable

observer is stable

error dynamics too!

Introduce:

$\tilde{y}(t) = y(t) - \hat{y}(t)$ - output error

$\tilde{x}(t) = x(t) - \hat{x}(t)$ - state reconst. error

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t)$$

Given $\tilde{x}_0 = x_0 - \hat{x}_0 \neq 0$

$$\tilde{x}(t) = e^{(A-KC)t}\tilde{x}_0$$

$\tilde{x}_\infty = 0 \Rightarrow x_\infty = \hat{x}_\infty$

unbiased

$A-KC$ stable

How to design K?

K can be obtained by eigenvalue assignment

Theorem (observer design)

Given a (full state) observeable LTI state-space model with open-loop eigenvalues $\{p_1, p_2, \dots p_n\}$ and $\{a_1, a_2, \dots, a_n\}$ stands for the characteristics polynomial coeff.

Then

$$\tilde{K} = T \cdot K = T \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$$

with $k_i = \bar{a}_i - a_i \quad \forall \; i = 1 \dots n$ where $\{\bar{a}_1, \dots \bar{a}_n\}$ / $\{\bar{p}_1, \dots \bar{p}_n\}$ refers to the closed-loop observer $(A-KC)$. Coeffs and eigenvalues (genetic)

T transform the state space to observable canonical form.

E.g. $G(s) = \dfrac{b_0}{s^2 + a_2}$, $b_0 = 1$, $a_2 = -1$

Find k such that the observers eigenvalues are allocated to $\bar{p}_1 = \bar{p}_2 = -10$.

step 1) Create the observeable canonical form (from $G(s)$)

step 2) Apply theorem

1)

$$A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ -a_2 & & & \vdots \\ \vdots & & & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix} ; \quad C = [1\, 0 \cdots 0]$$

$$n = 2 \Rightarrow A = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} ; \quad C = [1 \;\; 0]$$

$$\left.\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ +1 & 0 \end{bmatrix} \\ B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ C &= [1 \;\; 0] \end{aligned}\right\}$$

Open-loop system model in observable form

2) Open-loop eigenvalue & coeff.

$$\det(I\lambda - A) = \det \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

$$p_1 = -1 , \quad p_2 = 1$$

$$a_1 = 0 , \quad a_2 = -1$$

The closed-loop observer

$$\det(\lambda I - A + KC) = (\lambda + 10)^2 = (\lambda - p_1)(\lambda - p_2)$$

design condition

$$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$(\lambda + 10)^2 = \lambda^2 + 20\lambda + 100 \Rightarrow \begin{aligned} \bar{a}_1 &= 20 \\ \bar{a}_2 &= 100 \end{aligned}$$

$$\left.\begin{aligned} k_1 &= \bar{a}_1 - a_1 = 20 \\ k_2 &= \bar{a}_2 - a_2 = 101 \end{aligned}\right\} \Rightarrow K = \begin{bmatrix} 20 \\ 101 \end{bmatrix}$$

Remark: In a stable way by using input $u(t)$ and output $y(t)$ we designed a stable observer!

The original model is still unstable! (no control)

Let us use $\hat{x}(t)$ to create $u(t) = -L\hat{x}(t)$

2) State observer + state feedback (control)

Use $\hat{x}(t)$ to control the model

Theorem (principle of separation)

Given a minimal order LTI state-space model, feedback gains L, (control) and K (observer) can separately be designed.

(observable and controllable at the same time)

Model: $D \equiv 0$

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}\right\}$$

Policy: $u(t) = -L\hat{x}(t) + L_r r(t)$

Observer: $\dot{\hat{x}}(t) = (A-KC)\hat{x}(t) + Bu + Ky$

$\hat{y}(t) = C\hat{x}(t)$

Closed-loop (controller & observer)

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) - BL\hat{x}(t) + BL_r r(t) \\ y(t) &= Cx(t) \end{aligned}\right\} \pm BLx(t)$$

$-L\hat{x}(t) + L_r r(t)$

$$\dot{x}(t) = (A-BL)x(t) + BL\underbrace{(x(t) - \hat{x}(t))}_{\tilde{x}(t)} + BL_r r(t)$$

$$\dot{\tilde{x}}(t) = (A-KC)\tilde{x}(t)$$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} (A-BL) & BL \\ 0 & (A-KC) \end{bmatrix}}_{\substack{\mathcal{A} \\ \text{closed-loop} \\ \text{dynamics}}} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} + \begin{bmatrix} BL_r \\ 0 \end{bmatrix} r(t)$$

$$\det(\lambda I_{2n} - \mathcal{A}) = \det(\lambda I - A + BL)\det(\lambda I - A + KC)$$

Remark: closed-loop eigenvalues are the product of 2 separate det. conditions.
It says that the closed-loop stability (independently) depend on:
(i) stability of $A-BL$
(ii) —"— $A-KC$
$\Rightarrow$ separatly design stable, their combination remains stable. (Eigenvalue sep. principle).