14/5-18 Föreläsning 12

1) From output- to state-feedback design

2) Eigenvalue assignment (EA)

3) Optimal eigenvalue assignment (Linear quadratic control, (LQR))

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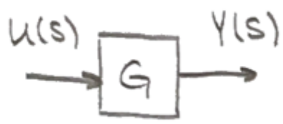
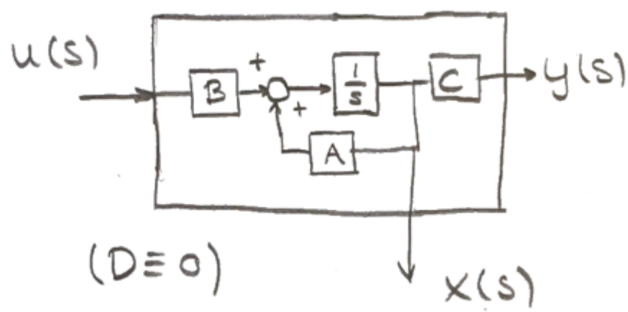
1). From output- to state-feedback design

Idea: Instead of y(t), let's use the state x(t) for u(t).

IO- model



State-space models



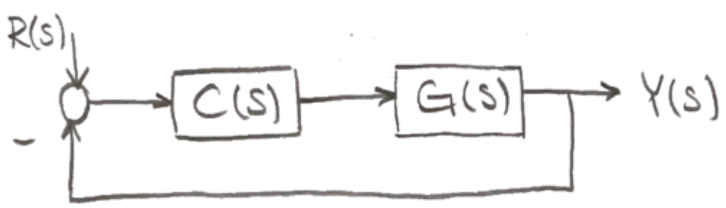
$$G(s) = C(sI - A)^{-1}B$$

Input to output vs. Internal

i) input to X(s)

ii) X(s) to Y(s)

IO- model based controller design (output-feedback)



We know how to find $C(s)$

$u(s) = C(s)(R(s) - Y(s))$

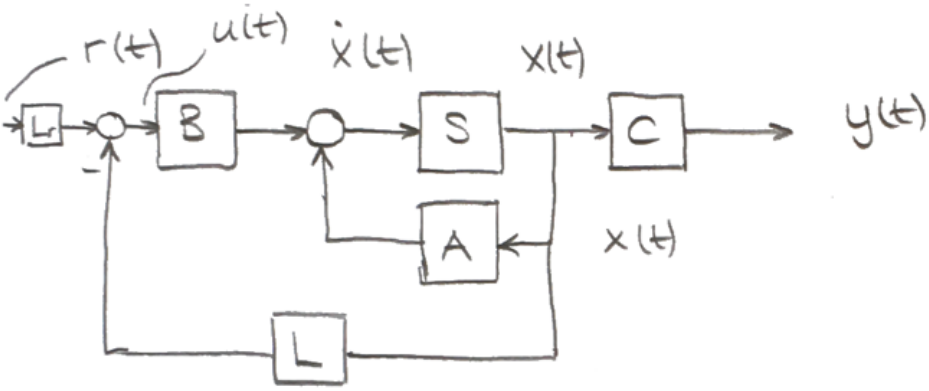
State feedback

$X(s)$ and feed it back (time domain)

$u(t) = -Lx(t) + L_r r(t)$

Condition: $x(t)$ measured.

$D \equiv 0$



$\dot{x}(t) = Ax(t) + Bu(t)$

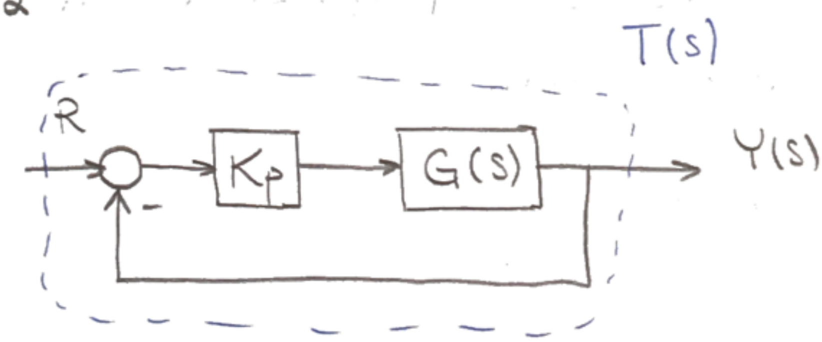
$y(t) = Cx(t) \quad (D \equiv 0)$

How design L, L_r?

What are the benefits of SF?

By using only $y(t)$ for controller design, some internal ($x(t)$) related info is hidden / not used.

Eg. $G(s) = \dfrac{1}{s^2 - s - 2}$, Find $C(s) = K_p$ that stabilizes $T(s)$.



$T(s) = Y(s)/R(s)$

comp. sens. fcn.

$$T(s) = \frac{L(s)}{1+L(s)} \quad ; \quad L(s) = G(s)K_p$$

$$T(s) = \frac{\frac{K_p}{s^2-s-2}}{1 + K_p/(s^2-s-2)} = \frac{K_p}{s^2-s-2+K_p} \quad , \text{ stable?}$$

Need for checking the poles of $T(s)$

$s^2 - s - 2 + K_p = 0$

$$p_{1,2} = \frac{-(-1) \pm \sqrt{1-4(K_p-2)}}{2}$$

, $\forall$ K_p we will have 1 pole $Re(p) > 0$

$\Rightarrow$ unstable.

The controller structure is not enough. (endast en parameter influeras utav K_p, inte $(-s)$)

$\Rightarrow$ We need a PD-controller!

$C(s) = K_p(1 + T_d \cdot s)$

$T_d \longrightarrow \quad s^2 + (K_p T_d - 1)s + K_p - 2 = 0$

We now influence both coefficients in the characteristic polynomial and hence we can stabilize. What if we have

$$G(s) = b_0 \frac{\prod_{j=1}^{m}(s-z_j)}{\prod_{i=1}^{n}(s-p_i)}$$

$\Rightarrow$ That claims nth order PD-term for stabilization.

In time domain:

$$\sum_{i=0}^{n} a_i \frac{d^{(n-i)} y(t)}{dt^{(n-i)}} = \sum_{j=0}^{m} b_j \frac{d^{(m-j)} u}{dt^{(m-j)}}$$

$a_0 = 1$ (monic polynomial)

Introduce

$$x(t) = \underbrace{\begin{bmatrix} \frac{dy^{(n-1)}}{dt^{(n-1)}} \\ \vdots \\ y(t) \end{bmatrix}}_{n \text{ elements}}$$

$x(t) \in \mathbb{R}^n$

$$\dot{x}(t) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} x(t) + \overset{m=0}{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} u(t)$$

$y(t) = [0 \cdots 0\ 1]\, x(t)$

What if $x(t)$ is measured?

Let's use it

2) Eigenvalue assignment

E.g. $\ddot{y}(t) - \dot{y}(t) - 2y(t) = u \quad \mathcal{L}^{-1}\{Y(s) = G(s)U(s)\}$

$$x(t) = \begin{bmatrix} \dot{y}(t) \\ y(t) \end{bmatrix}$$

$\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, $D \equiv 0$

$$A = \begin{bmatrix} -(\overset{a_1}{-1}) & -(\overset{a_2}{-2}) \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Let's use statespace for stabilization

When can we say a state-space form i stable

$\dot{x}(t) = Ax(t) + Bu(t)$

$A \in \mathbb{R}^{n \times n}$

stability

$\det(\lambda I - A)^{-1} = 0$ eigenvalues of A

$Re(\lambda_i) < 0 \quad \forall i$ then a state-space model is asymptotically stable.

$\Rightarrow$ If eigenvalues are unstable, let's move them to stable domain. (assign stable eigenvalues)

Assume $x(t)$ is measurable

$$u(t) = -L \cdot \underset{\substack{\uparrow \\ \text{state}}}{x(t)} + L_r \cdot \underset{\substack{\uparrow \\ \text{reference}}}{r(t)} \quad ; \quad L \in \mathbb{R}^{1 \times n} ; \; L_r \in \mathbb{R}$$

How to find L, L_r?

$\dim(X) = n$

Two degree of freedom: L, L_r

DOF 1) L has to stabilize the state-space Closed-loop

$$\dot{x}(t) = A\,x(t) + B(-L\,x(t) + L_r\,r(t))$$

$$\dot{x}(t) = \underbrace{(A - BL)}_{\text{closed-loop state-space matrix}}\,x(t) + BL_r \cdot r(t) \quad *$$

$$y(t) = C\,x(t) + D\,u(t) = (C - DL)\,x(t) + DL_r\,r(t)$$

direct feedthrough active

L has to be chosen such that

$\det(I \cdot \lambda - (A - BL)) \quad \forall\, i \quad Re(\lambda) < 0$

DOF 2) $L_r = ?$

L_r is a shaper, changes $r(t)$

$\Rightarrow$ use to shape $r_\infty = y_\infty$

Match asymptotical tracking of y_∞ over r_∞.

Steady state tracking

$$0 = (A - BL)\,x_\infty + BL_r \cdot r_\infty$$

$$y_\infty = (C - DL)\,x_\infty + DL_r \cdot r_\infty$$

Given L stabilizing $(A - BL)$

$$x_\infty = -(A - BL)^{-1} \cdot BL_r \cdot r_\infty$$

$$y_\infty = (((C - DL)(BL - A)^{-1}B + D)L_r \cdot r_\infty)$$

$y_\infty = r_\infty$

$$\frac{1}{L_r} = (C-DL)(BL-A)B + D$$

$\Rightarrow$ first L, then L_r

Algoritm (eigenvalue assignment)

Given (A, B, C, D)

step 1) Check controllability, i.e. (A,B) is controllable. Full state controllability is a must. (Kalman rank condition).

step 2) If controllable $(A, B, C, D) \underset{T^{-1}}{\overset{T}{\rightleftarrows}} (A_c, B_c, C_c, D_c)$ i.e. transform it into 'canonical' form.

step 3) for OL:

$\det(\lambda I - A) = (\lambda - p_1)(\lambda - p_2) \dots (\lambda - p_n) =$

$= \lambda^n + a_1 \lambda^{n-1} \dots + a_n$

We know the closed-loop locations

$\det(\lambda I - A + BL) = (\lambda - \tilde{p}_1)(\lambda - \tilde{p}_2) \dots (\lambda - \tilde{p}_n)$ $\leftarrow$ CL

again $\underbrace{\begin{matrix}\{p_1 \dots p_n\} \\ \{\tilde{p}_1 \dots \tilde{p}_n\}\end{matrix}}_{\text{are known}}$

$L = [l_1, l_2 \dots l_n] \quad \forall\, i = 1 \dots n$

$\boxed{l_i = \tilde{a}_i - a_i}$ — open loop coeffs.

closed loop coeffs.

(reson why we need it in canonical form)

So, given a controller canonical form

$$u(t) = L x(t) = -[l_1 \; \dots \; l_n] \, x(t)$$

Closed-loop

$$\dot{x}(t) = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_n \\ 1 & \ddots & & 0 \\ & & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} x(t) - \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} \underbrace{[l_1 \; l_2 \; \dots \; l_n] \, x(t)}_{-u(t)}$$

$$\dot{x}(t) = \begin{bmatrix} -a_1 - l_1 & -a_2 - l_2 & \dots & -a_n - l_n \\ 1 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix} x(t)$$

$r \equiv 0$

$$\det(\lambda I - A + BL) = \lambda^n + \underbrace{(a_1 + l_1)}_{\tilde{a}_1 = a_1 + \tilde{a}_1 - a_1} \lambda^{n-1} + \dots a_n + l_n$$

Remark: eigenvalue allocation / poleplacement / Akerman

E.g.

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

Find L, such that $\tilde{p}_1 = \tilde{p}_2 = -1$

$L = [l_1 \; l_2]$

$u(t) = -L x(t) = -l_1 x_1(t) - l_2 x_2(t)$

$$l_1 = \tilde{a}_1 - a_1$$
$$l_2 = \tilde{a}_2 - a_2$$

$\{a_1, a_2\}$? coeff. of the open-loop model's characteristics polynomial.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} = \underbrace{\lambda^2 + 1}$$

$a_1 = 0$ $\quad$ $a_2 = 1$

Closed loop

$$\det(\lambda I - A + BL) = (\lambda - \tilde{p}_1)(\lambda - \tilde{p}_2) = (\lambda + 1)^2 = \underbrace{\lambda^2 + 2\lambda + 1}$$

$$L = [l_1 \ \ l_2] = [2 \ \ 0]$$

$\tilde{a}_1 = 2$ $\quad$ $\tilde{a}_2 = 1$

$\Rightarrow u(t) = -2x_1(t)$

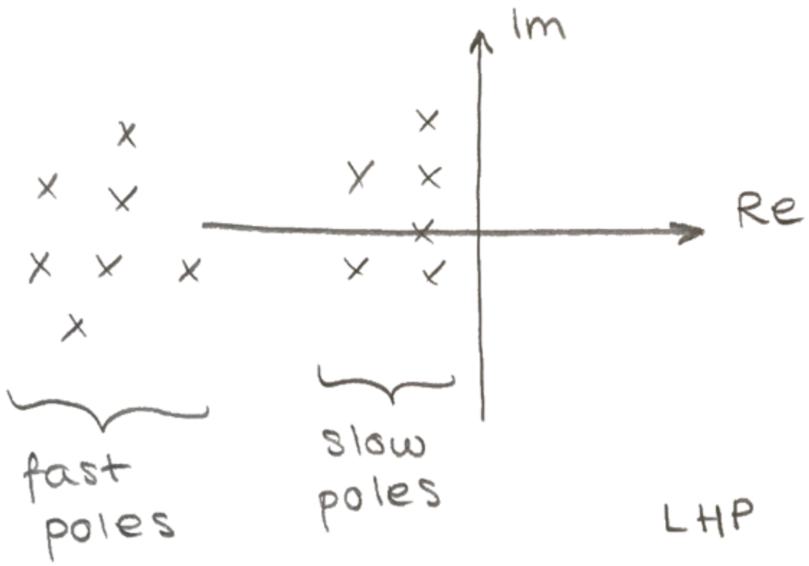
Theorem (EA)

Given a (full state) controllable LTI state-space model with open/closed-loop poles $\{p_1 \dots p_n\} / \{\tilde{p}_1 \dots \tilde{p}_n\}$, then

$L = L_c T = [l_1 \dots l_n] T$ will assign the closed-loop eigenvalues to $\{\tilde{p}_1, \dots \tilde{p}_n\}$. T is the similarity state transformation matrix $T \in \mathbb{R}^{n \times n}$

3) Optimal EA

only a region of eigenvalue



How to find the optimal pole location?