7/5-18 Föreläsning 11

1) Robust stability
2) Uncertainty
   A) Additive
   B) Multiplicative
3) Small gain theorem
4 Bode sensitivity

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1) Robust stability

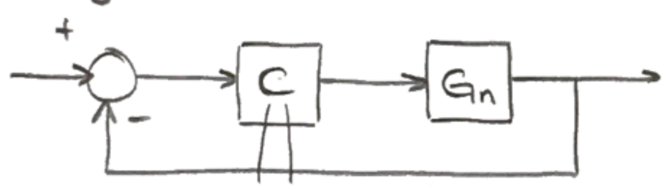
Problem: We usually do not know the "real" system model.

We only know: $G_n(s)$ ← nominal simplified model.

$G(s) \sim G_n(s)$

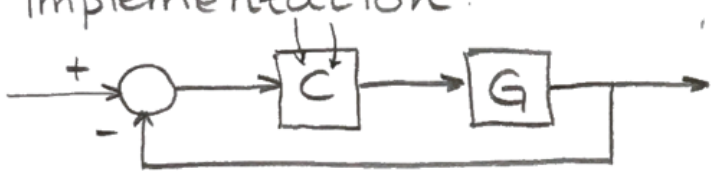
complex — simplified (e.g first principle modeling concepts)

Design:



$C(s)$, $G_n(s)$ are avaliable

Implementation:



Is this stable too?

Def. (robust stability)

The output feedback controller $C(s)$ is called robustly stable if it stabilizes both the nominal $\boxed{\left(T_n(s) = \frac{L(s)}{1+L(s)}\right)}$ and the real $\boxed{\left(T(s) = \frac{1}{1+L(s)}\right)}$

Robustly stable $C(s)$ will ensure stability of

$$T(s) = \frac{C(s)G(s)}{1+C(s)G(s)} \quad ; \quad T_n(s) = \frac{L_n(s)}{1+L_n(s)} = \frac{C(s)G_n(s)}{1+G_n(s)C(s)}$$

Complex model $G(s)$?

Order of $G(s)$ (n, m) is larger compared to $G_n(s)$

2) Robust stability tests

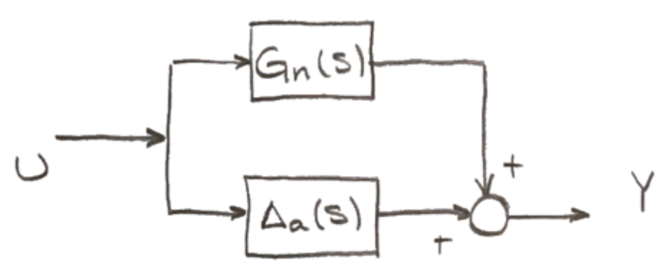
Problem: $C(s)$ is robust?

Solution: setup constraints to $C(s)$ (inequality constr.) To reach it, we need info. on uncertainty.

Uncertainty, model-mismatch is the difference between $G_n(s)$ and $G(s)$.

A) Additive

$$G(s) \triangleq G_n(s) + \Delta_a(s)$$

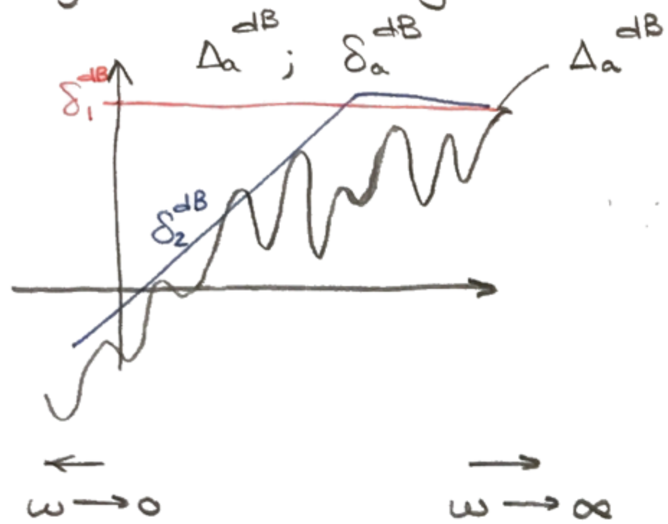


Additive uncertainty is reformed as absolut one.

Usually $\Delta_a(s)$ is unknown. However $\delta_a(s) > \Delta_a(s)$

dynamic overbound on $\Delta_a(s)$

e.g. Bode magnitude



δ_1 = constant

δ_a = dynamic

Δ_a at low frequency is low.

Based on $G_n(s)$, $\delta_a(s)$, $C(s)$ we can decide on robust stability

Theorem (additive test)

Given $G_n(s)$, the nominal/simplified model, $\delta_a(s)$ the overbound of addtive uncertainty, $C(s)$ is robustly stable if:

$$|\delta_a(s)| < \left| \frac{1 + G_n(s)C(s)}{C(s)} \right|$$

$\Longleftrightarrow$

nominal sensitivity

$$\left| \frac{1}{\delta_a(s)} \right| > \left| C(s) \cdot S_n(s) \right| \quad , \quad S_n(s) = \frac{1}{1 + L_n(s)}$$

Proof: Nyquist stability theory, stability margin:

$$\min_s |L_n(s) - (-1)| = \min_s |L_n(s) + 1|$$

Nominal loop $L_n(s)$ is controlled/stabilized with margin. How about $G(s)$? "real"

$L(s) = C(s)G(s)$

$L(s) - L_n(s) = C(s)(G(s) - G_n(s)) = C(s)\Delta_a(s)$

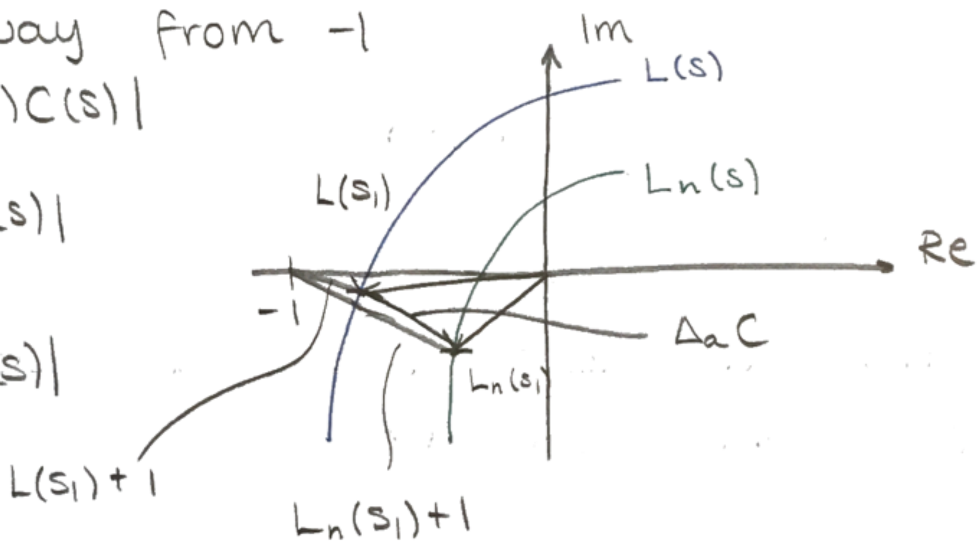
Now, diff between $L(s)$ and $L_n(s)$ $\Delta_a \cdot C(s)$

$|L_n(s)+1|$ is away from -1

$|L_n(s)+1| > |\Delta_a(s)C(s)|$

$\Rightarrow |\Delta_a(s)| < |\delta_a(s)|$

$$\frac{|L_n(s)+1|}{|C(s)|} > |\delta_a(s)|$$

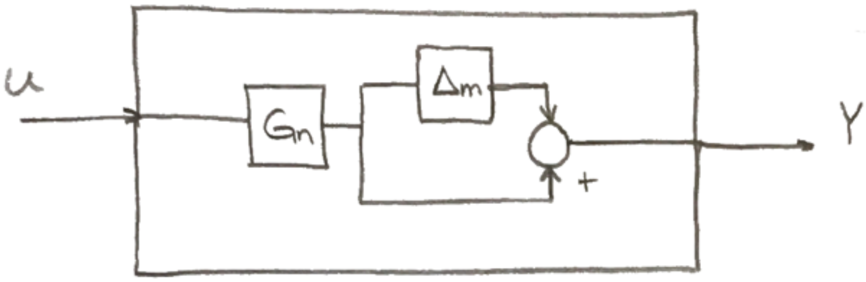


$L(s_1)$ / $L_n(s_1)$ stability margin for $\begin{cases}\text{real} \\ \text{nominal}\end{cases}$ models

Remark: Robust stability test is a lower bound on the nominal stability margin $|\Delta_a(s) \cdot C(s)|$ is the lower bound.

B) Multiplicative

$G(s) \triangleq (1+\Delta_m(s)) \cdot G_n(s)$



$$\Delta_m(s) = \frac{G(s) - G_n(s)}{G_n(s)} \qquad [\%]$$

Multiplicative uncertainty is a relative metric.

Remark :
$$\Delta_m = \frac{\Delta_a(s)}{G_n(s)}$$

multiplicative and additive c structure is the same.

Theorem (multiplicative robust stability test)

Given the nominal/simplified $G_n(s)$ and overbond on the multip. uncertainty fcn. $\delta_m(s) > \Delta_m(s)$, $C(s)$ is robustly stabilizing if

$$|\delta_m(s)| < \left| \frac{1 + L_n(s)}{L_n(s)} \right|$$

Equivalenty,
$$\frac{1}{|\delta_m(s)|} > |T_n(s)|$$

nominal,
complementary transfer fcn.

Remark: Multiplicative robust test is a lowerbound on the inverse of $T_n(s)$ the lowerbond is $|\delta_m(s)|$

eg. We will enforce robust stability tests in frequency domain

$$G_n(i\omega) = \frac{3}{i\omega + 1}$$

$$C(i\omega) = K_p \quad , \quad K_p = \{10 ; 1 ; 0{,}2\}$$

Which controller is robust against $\delta_m(i\omega) = i\omega/(0{,}4i\omega + 1)$

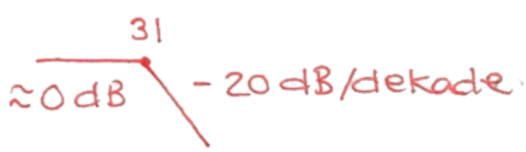
Recall the multiplicative robust tests:

$$\frac{1}{|\delta_m(i\omega)|} > |T_n(i\omega)|$$

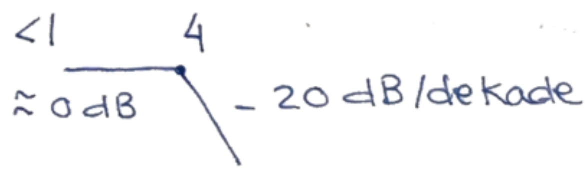
Let's find 3 $T_n(i\omega)$ with the 3 controllers

$$T_n(i\omega) = \frac{L_n(i\omega)}{1 + L_n(i\omega)} = \frac{K_p(3/(i\omega+1))}{1 + K_p(3/(i\omega+1))} = \frac{3K_p}{3K_p + i\omega + 1}$$

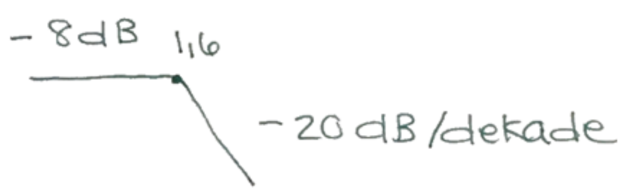
$\underline{K_p = 10}$) $T_n(i\omega) = \frac{30}{31} \cdot \frac{1}{\frac{i\omega}{31} + 1}$



$\underline{K_p = 1}$) $T_n(i\omega) = \frac{3}{4} \cdot \frac{1}{\frac{i\omega}{4} + 1}$



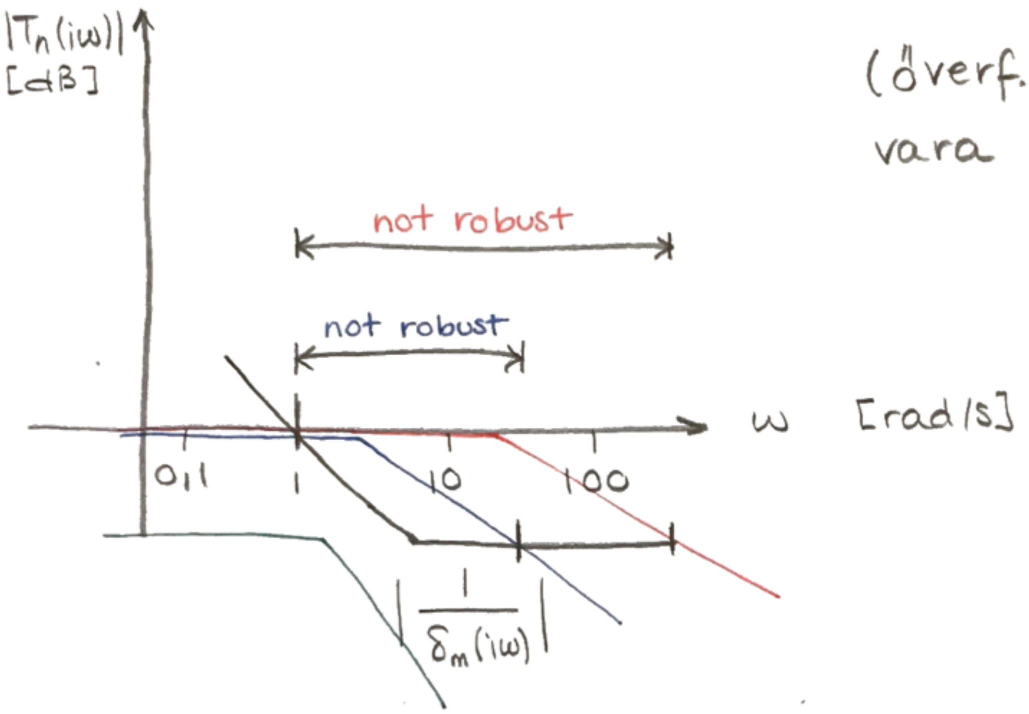
$\underline{K_p = 0{,}2}$) $T_n(i\omega) = \frac{0{,}6}{1{,}6} \cdot \frac{1}{\frac{i\omega}{1{,}6} + 1}$



$$\frac{1}{\delta_m(i\omega)} = \frac{0{,}4i\omega + 1}{i\omega}$$

For every robust test we only plot magnitude!

(överf. fkn måste vara minst)



We can test against robustness controllers.

3) Small gain theorem

Multiplicative test

$$\underbrace{|\delta_m|}_{\text{gain 1}}\underbrace{|T_n|}_{\text{gain 2}} < 1$$

Additive

$$\underbrace{|\delta_a(s)|}_{\text{gain 1}}\underbrace{|C(s)S_n(s)|}_{\text{gain 2}} < 1$$

Product of gains (magnitude) has to be smaller than 1. Then the closed-loop is input-output stable. (Definition).

Remark: All of the gains product have to obey SGT.

Generalization of BIBO concept.

4) Bode sensitivity

There are limitation controller design.

e.g $S(s) + T(s) = 1$

rejection tracking

Theorem

Assume $L(i\omega) = G(i\omega) \cdot C(i\omega)$ has n_p poles at RHP (unstable poles)

$K = \lim_{\omega \to \infty} i\omega L(i\omega)$ $\exists$.

(striktly proper $L(i\omega)$)

$$\int_0^\infty \ln |S(i\omega)| \, d\omega = \begin{cases} \pi \sum_{i=1}^{n_p} \mathrm{Re}(p_i) & (i) \\ \times - K\frac{\pi}{2} & (ii) \end{cases}$$

(i) $n - m > 1$

(ii) $n - m = 1$

If $n_p = 0$

$$\int_0^\infty \ln |S(i\omega)| \, d\omega = 0$$