2/5-18 # Föreläsning 10

1) Non-minimal phase system models
   A. Unstable zeros
   B. Delay
   C. Delay compensation
2) Robust stability

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1) Non-minimal phase system models

Problem: Specific classes of dynamic system models that behaves both in time and frequency domain "odd".

E.g. over/under steered bikes/cars; rear steered bicycle.

Def. (minimal phase lag)

Every rational, minimum phase frequency fcn. $G(i\omega) = b_0 \dfrac{\prod_{l=1}^{m}(i\omega - z_l)}{\prod_{j=1}^{n}(i\omega - p_j)}$ has a phase

lag $-\frac{\pi}{2}(n-m)$ where n/m are the number of poles and zeros.

E.g. $G(i\omega) = \dfrac{1}{i\omega + 1}$

phase lag: $-\frac{\pi}{2}(1-0) = -\frac{\pi}{2}$

(phase shift $\omega \rightarrow \infty$)

Non-minimum $G(i\omega)$ are shifting more than the $-\frac{\pi}{2}(n-m)$

A. "UNSTABLE" ZEROS

$Re(z_\ell) > 0$

B. TIME DELAY

$y(t) = u(t-\tau) \quad \tau > 0$

E.g $\quad G(s) = \dfrac{3-s}{(s+1)(s+5)}$

Step responce!

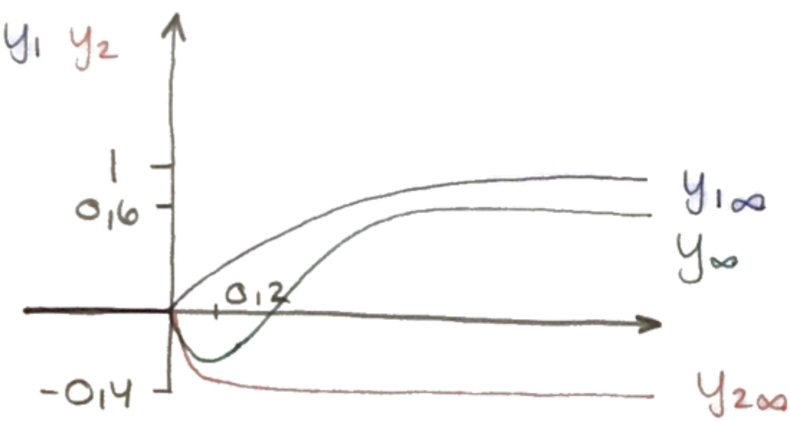
$$G(s) = \frac{1}{s+1} - \frac{2}{s+5} = \frac{1}{\frac{s}{1}+1} - \frac{0{,}4}{\frac{s}{5}+1}$$

(polynomial division)

time constant

$y_1 \quad y_2$

$y_1 = (1-e^{t}) \;, \quad y_2 = -0{,}4(1-e^{t/1/5})$



step response $y(t)$

($y_{2\infty}$ går mot oändligheten 5ggr snabbare än y_1)

REMARK: TIME domain, transient property of non-minimum phase model is to start with <u>opposite</u> direction.

Frequency domain: phase change is
$-(\pi/2)(n-m)$

E.g. $$G(i\omega) = \frac{1}{i\omega+1}\left(\frac{1-\tau i\omega}{1+\tau i\omega}\right)^k$$

$(k = 0, 1, 2, \ldots)$

Magnitude and phase behaviour?

unstable zero $\Rightarrow$ NMP

<u>Magnitude</u>

$$|G(i\omega)|^2 = \underbrace{\frac{1}{1+\omega^2}}_{MP}\underbrace{\left(\frac{1+\cancel{(-\tau\omega)^2}}{1+\cancel{(\tau\omega)^2}}\right)^k}_{n-MP} = \frac{1}{1+\omega^2}$$

Same as for a first order lag. Unstable zero does not influence the magnitude plot.

<u>Phase</u>

$$\varphi(\omega) = \underbrace{\varphi_1(\omega)}_{MP} + \underbrace{\varphi_2(\omega)}_{n-MP} = -\tan^{-1}(\omega) - 2k\tan^{-1}(\omega\tau)$$

$$G_1(i\omega) = \frac{1}{i\omega+1}$$

$$G_1(i\omega) = \frac{1}{\omega^2+1} - \frac{i\omega}{\omega^2+1}$$

$$\tan^{-1}\left(\frac{Im(\ )}{Re(\ )}\right) = \varphi_1(\omega)$$

$$= -\tan^{-1}(\omega)$$

HW: $k=1$

$$G_2(i\omega) = \frac{1}{(i\omega+1)} \cdot \frac{(1-\tau i\omega)}{(1+\tau i\omega)}$$

$\tau > 0$

REMARK: Phase shift is smaller with unstable zeros compared to their minimum phase counter-part. Shifts phase more than $-\frac{\pi}{2}(n-m)$

They require specific handling when using phase margins (Bode compensation)

- <u>Disturbance rejection?</u>

$$\frac{Y(s)}{D(s)} = S(s) = \frac{1}{L(s)+1} = \frac{1}{C(s)G(s)+1} = 1$$

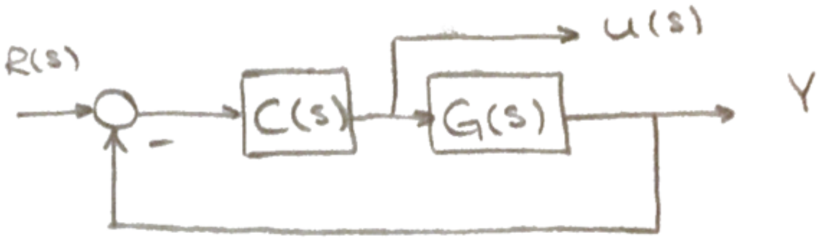
assume $G(s)\Big|_{s=z_l} = 0$, $Re(z_l) > 0$

at zeros, no disturbance rejections

- <u>Zero cancelling control policy?</u>

$$C(s)G(s) = \underbrace{\frac{\tilde{b}(s)}{\tilde{a}(s)\cancel{(s-z_l)}}}_{\text{controller}} \cdot \underbrace{\frac{\cancel{(s-z_l)}b(s)}{a(s)}}_{G(s)}$$

$\Rightarrow$ No good solution C(s) becomes unstable!
E.g. certain transfer fcn may become unstable to



Any transfer fcn. from R(s) to u(s)

$$\frac{U(s)}{R(s)} = \frac{C(s)}{1 + C(s)G(s)}$$

unstable

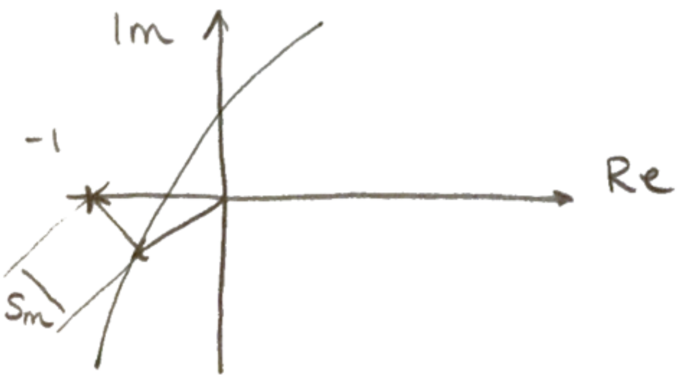
unstable loop!

Limitation with unstable zeros (maximum modulus theorem).

Theorem: If $G(i\omega)$ has one unstable zero (z) and pole (p), then the stability margin gets a lower bound

$$S_m = \min_{\omega} |L(\omega) - (-1)| = \max_{\omega} |S(i\omega)| \quad , \quad S_m \geq \left| \frac{z+p}{z-p} \right|$$

⇒ Conclusion: hard to control a plant with large p, z. Any small uncertainty will destab. it.



B) TIME DELAY

Pure time delay is a NMP system model.

E. g. chemical processes (paper mill / steel), temp. control

E. g.

$$G(i\omega) = \frac{1}{i\omega+1} e^{-i\omega\tau} \qquad \tau > 0$$

Bode diagram? Magnitude / phase

$$|G(i\omega)|^2 = \frac{1}{1+\omega^2} \cdot 1$$

alltid

⇒ magnitude of a delayed model is the same as the delay-free answer

Phase

$$\varphi(\omega) = \underbrace{\varphi_1(\omega)}_{\text{f.o - lag}} + \underbrace{\varphi_2(\omega)}_{\text{delay}} = -\arctan(\omega) - \tau \cdot \omega$$

$\omega \to \infty \qquad \varphi(\omega) \to -\infty$

Delayed system model have infinite phase lag.

⇒ They are not minimum phase models

We can approx. $e^{-i\omega\tau}$

1) Truncation based approx. (by proper frequency fcn.)

first order

$$e^{-i\omega\tau} = \frac{e^{-i\omega\tau/2}}{e^{i\omega\tau/2}} \simeq \frac{1 - \frac{i\tau\omega}{2}}{1 + \frac{i\tau\omega}{2}}$$

We have unstable zero
$\Rightarrow$ NMP approx.

REMARK:

⊕ Proper approx. of an irrational $e^{-i\omega\tau}$

⊖ Unstable zero

⊖ Order is larger than 5 approx is unstable.

2) Padé - approx.

$$e^{-i\omega\tau} \approx \frac{p(\omega\tau)}{q(\omega\tau)}$$

, q, p are rational polynomials
$q(\ )$ is stable.

c) Delay compensation

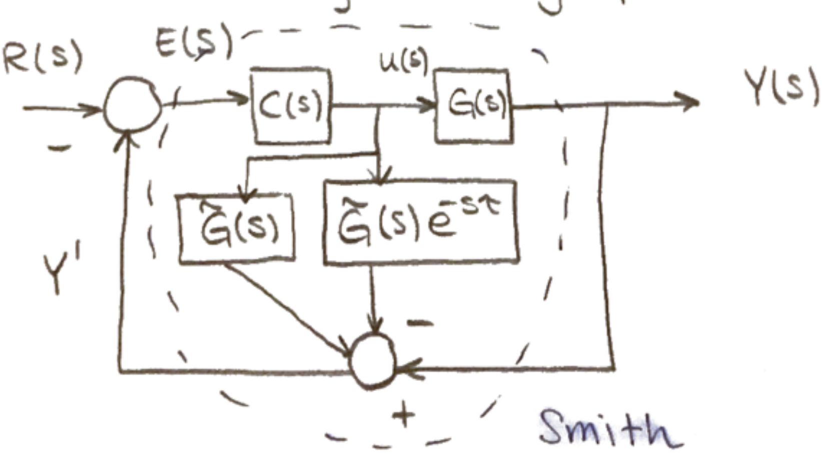
Idea

$$\underbrace{G(s) = \tilde{G}(s) \cdot \underbrace{e^{-s\tau}}_{\substack{\text{delay}\\ \tau>0}}}_{\text{known}}$$

, $\tilde{G}(s)$ - delay-free transf. fcn.

Smith predictor (50's)

- try compensate for delays action
- control only delay-free



$\tau > 0$ and $\tilde{G}(s)$ is (approx) know use the delayfree + delayed prediction for the output.

$$E(s) = R(s) - Y'(s) = R(s) - (\underbrace{\cancel{G(s)}u(s)}_{Y(s)} - \tilde{\cancel{G}}(s)e^{-s\tau} \cdot u(s) +$$

$$+ \tilde{G}(s)u(s))$$

$$\tilde{G}(s)e^{-s\tau} = G(s)$$
$$E(s) = R(s) - \tilde{G}(s)u(s)$$

$$u(s) = C(s) \cdot E(s) = C(s)R(s) - C(s)\tilde{G}(s)u(s)$$

closed the loop with a controller

$$T(s) = \frac{Y(s)}{R(s)} = \frac{C(s)}{1 + \tilde{G}(s)C(s)}$$

The closed-loop is delay-free
Birth of model predictive control (MPC)

2) Robust Stability
How much uncertainty can we tolerate?

$$\frac{Y(s)}{U(s)} = G(s) \Rightarrow Y(s) - G(s) \cdot u(s) \cong 0$$

If $G(s)$, the model is only a approx.
$(Y(s) \approx G(s)u(s))$
How can we implement a controller that does not know the real system behaviour?

Make sure it is robust!

problem statement:

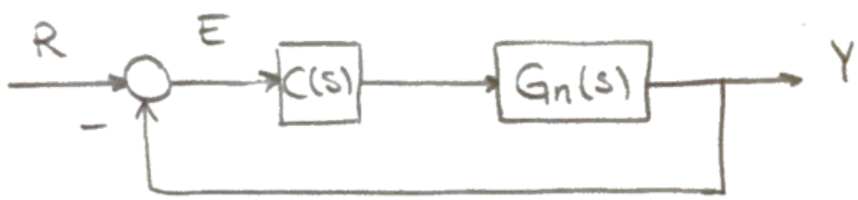
$G_n(s)$ - nominal transfer function

simple

$G(s)$ - complex describing reality

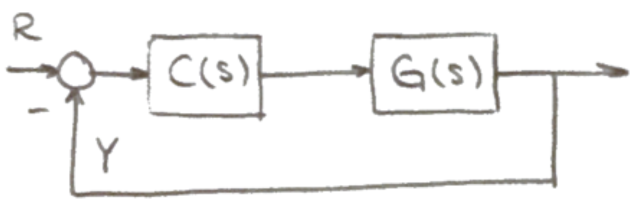
complex

Nominal $G_n(s)$ is available for design $C(s)$



We design $C(s)$ with $G_n(s)$ at hand, eg. Balduino

Implementation



What are the conditions that $C(s)$ will stabilize both $L_n(s)$ and $L(s)$?